Strassen algorithm. Advanced optimization for Intel(R) Xeon Phi(TM)

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Strassen algorithm

The usual number of scalar operations (i.e., the total number of additions and multiplications) required to perform matrix multiplication is

$$M(n) = 2n^3 - n^2$$

However, Strassen (1969) discovered how to multiply two matrices in

$$S(n) = 7n^{\log_2 7} - n^2 \approx 7n^{2.81} - n^2$$

References

- Bailey, D., Lee, K., Simon, H.: Using Strassen's Algorithm to Accelerate the Solution of Linear Systems. J. of Supercomputing 4(4), 357-371 (1991)
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- ElGindy, H., Ferizis, G.: On Improving the Memory Access Patterns During The Execution of Strassen's Matrix Multiplication Algorithm. ACSC '04 Proc. of the 27th Australasian conf. on Computer science – Vol. 26, 109-115 (2004)
- Li, J., Ranka, S., Sahni, S.: Strassen's Matrix Multiplication on GPUs. ICPADS '11 Proc. of the 2011 IEEE 17th International Conf. on Parallel and Distributed Systems, 157-164 (2011)
- Lipshitz, B., Ballard, G., Demmel, J., Schwartz, O.: Communication-avoiding parallel strassen: implementation and performance. SC '12 Proc. of the International Conf. on High Performance Computing, Networking, Storage and Analysis, Art. 101 (2012)
- Heinecke, A.: Evaluation of DGEMM Implementation on Intel Xeon Phi Coprocessor. J. of Computers, Vol 9, № 7 (2014), 1566-1571 (2014)
- Intel® Threading Challenge 2009. 2.1 Strassen Matrix Multiplication

Strassen algorithm

$$C = A * B = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} * \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} =$$

$$= \begin{bmatrix} S_1 + S_2 + S_3 - S_4 & S_4 + S_6 \\ S_3 + S_5 & S_1 - S_5 + S_6 + S_7 \end{bmatrix}$$

$$S_1 = (A_{00} - A_{11}) * (B_{00} + B_{11})$$
$$S_2 = (A_{01} - A_{11}) * (B_{10} + B_{11})$$
$$S_3 = A_{11} * (B_{10} + B_{00})$$
$$S_4 = (A_{00} + A_{01}) * B_{11}$$
$$S_5 = (A_{10} + A_{11}) * B_{00}$$
$$S_6 = A_{00} * (B_{01} - B_{11})$$
$$S_7 = (A_{10} - A_{00}) * (B_{00} - B_{01})$$

Connection to cluster

```
putty, winscp
ssh tornado.hpc.susu.ac.ru
user: userX
password: ******
```

```
http://supercomputer.susu.ru/users/support/
```

Serial version

- Folder 1
- build_x64.sh, build_mic.sh – building scripts
- matrix_strassen – wrappers for Strassen and Intel(R) MKL DGEMM
- matrix_strassen – Strassen implementation
- matrix_utils – matrix operations routines
- run – parsing command line, launch multiplication
- test_utils – testing execution time of matrix multiplication algorithms
- unit_tests – unit tests
- utils – memory routines

```
user1@login:~/1/src
$ ls
CUnit        matrix_mult.c      matrix_strassen.h  run.c         unit_tests.c
extra_libs   matrix_mult.h      matrix_utils.c     test_utils.c  utils.c
Makefile     matrix_strassen.c  matrix_utils.h     test_utils.h  utils.h
```

Verification

- Compare with Intel(R) MKL DGEMM

$$\varepsilon_{abs} = \max_{i,j \in 1..N} \left| a_{i,j} - b_{i,j} \right|$$

$$\varepsilon_{rel} = \max_{i,j \in 1..N} \frac{\left| a_{i,j} - b_{i,j} \right|}{\left| b_{i,j} \right|}$$

- Test succeeded if:

$$max(\varepsilon_{abs}, \varepsilon_{rel}) < 10^{-10}$$

Time measurement (test_utils.c)

- second/dsecnd - returns elapsed CPU time in seconds.

```
double run_method(MatrixMultiplicationMethod mm_method,
                  int n, double *A, double *B, double *C)
{
        int runs = 5;
        mm_method(n, A, B, C);

        double start_time = dsecnd();
        int i;
        for (i = 0; i < runs; ++i)
                mm_method(n, A, B, C);

        double end_time = dsecnd();
        double elapsed_time = (end_time - start_time) / runs;
        return elapsed_time;
}
```

matrix_strassen.c

```
void strassen_1( int n, double *A, int lda,
                        double *B, int ldb,
                        double *C, int ldc,
                                   int threshold)
{
      if (n <= threshold)
      {
            mkl_simple_ld(n, A, lda, B, ldb, C, ldc);
            return;
      }

…
```

matrix_strassen.c

```
int i, j;
for (i = 0; i < 2; ++i)
for (j = 0; j < 2; ++j)
{
      sub_a[i][j] = &A[i * n2 * lda + j * n2];
      sub_b[i][j] = &B[i * n2 * ldb + j * n2];
      sub_c[i][j] = &C[i * n2 * ldc + j * n2];
}
```

$$\begin{pmatrix} \boldsymbol{a} & a & \boldsymbol{a} & a \\ a & a & a & a \\ \boldsymbol{a} & a & \boldsymbol{a} & a \\ a & a & a & a \end{pmatrix}$$

matrix_strassen.c

$$M_6 = (A_{10} - A_{00}) * (B_{00} - B_{01})$$

```
…
matrix_sub_ld( n2, sub_a[1][0], lda,
                   sub_a[0][0], lda,
                                sub_c[0][1], ldc);

matrix_sum_ld( n2, sub_b[0][0], ldb,
                    sub_b[0][1], ldb,
                    sub_c[1][0], ldc);

strassen_1( n2, sub_c[0][1], ldc,
                sub_c[1][0], ldc,
                sub_c[1][1], ldc, threshold, n_threads); // m6

…
```

Intel(R) Math Kernel Library

$$C = \alpha \cdot op(A) + \beta \cdot op(B)$$

```
mkl_domatadd('R', 'N', 'N', n, n, 1.0,
                            A, lda, 1.0, B, ldb, C, ldc);
```

$$C = \alpha \cdot A \cdot B + \beta \cdot C$$

```
double alpha = 1.0;
double beta = 0.0;
cblas_dgemm( CblasRowMajor, CblasNoTrans, CblasNoTrans,
              n, n, n, alpha, a, lda, b, ldb, beta, c, ldc);
```

```
mkl_malloc, mkl_free, mkl_peak_mem_usage, ..
```

Build and run

```
./build_x64.sh
./build_mic.sh

chmod +x build_mic.sh – разрешить выполнение

salloc – выделить узел
squeue – просмотр выделенных узлов

ssh ps-micX – зайти на mic

exit – выйти с mic

scancel job_id – снять задачу
```

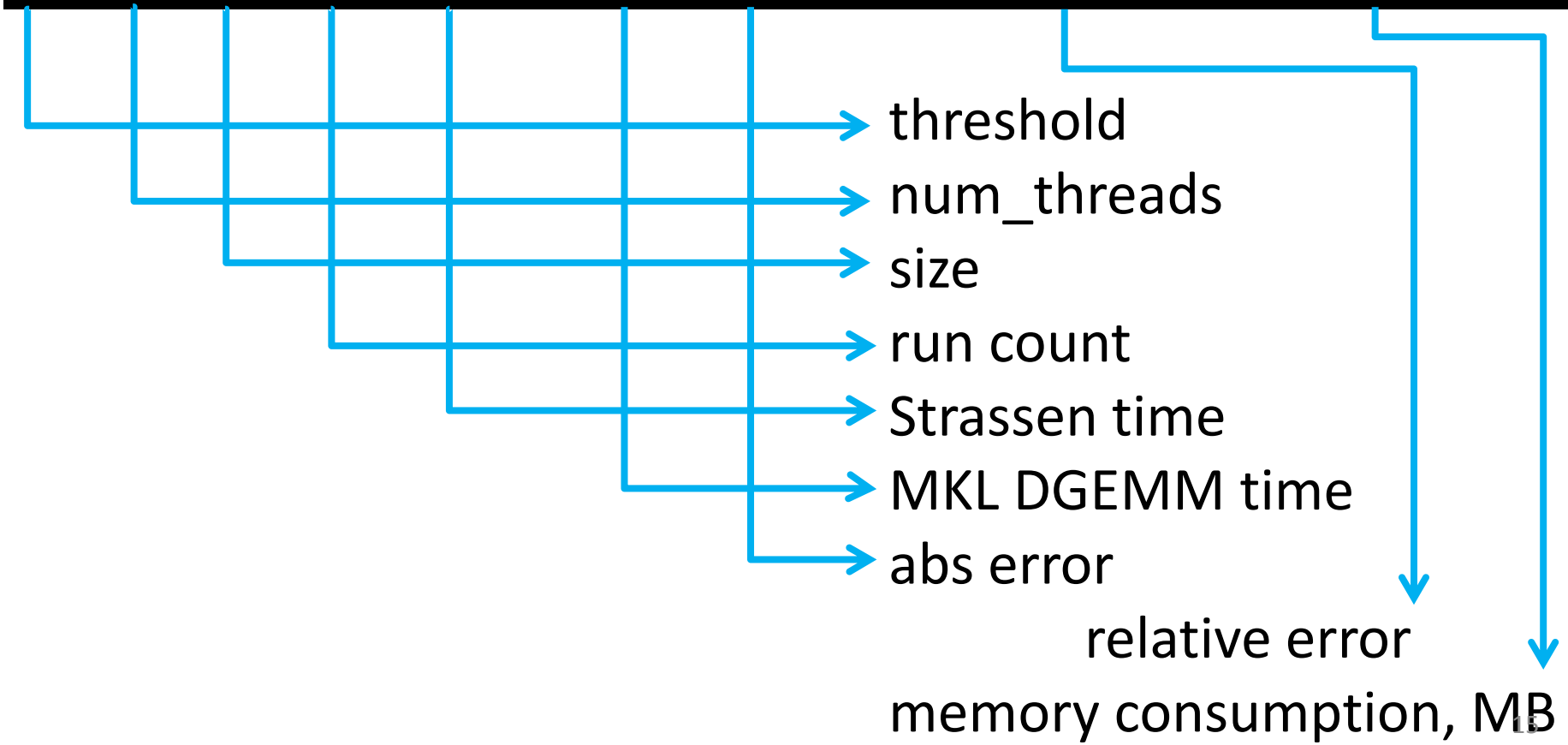
Build and run

```
./1/build_mic/run -t 512 -n 1 -s 4096 -r 1
./1/build_mic/run -t 256 -n 1 -s 4096 -r 1
./1/build_mic/run -t 1024 -n 1 -s 4096 -r 1
./1/build_mic/run -t 512 -n 60 -s 4096 -r 1
./1/build_mic/run -t 2048 -n 240 -s 8192 -r 1 -o
```

Max size ≈ 20000

Build and run

```
user1@ps-mic0:~/1/build_mic
$ ./1/build_mic/run -t 512 -n 60 -s 4096 -r 1
-sh: ./1/build_mic/run: No such file or directory
user1@ps-mic0:~/1/build_mic
$ ./run -t 512 -n 60 -s 4096 -r 1
512; 60; 4096; 1; 20.834; 0.534; 1.614353e-11; 1.544938e-14; 5.781524e+02;
```



Parallel Strassen algorithm

- Folder 2

```
CUnit        matrix_mult.c      matrix_strassen.h  run.c          unit_tests.c
extra_libs   matrix_mult.h      matrix_utils.c     test_utils.c   utils.c
Makefile     matrix_strassen.c  matrix_utils.h     test_utils.h   utils.h
```

Parallel Strassen algorithm

$$C = A * B = \begin{bmatrix} S_1 + S_2 + S_3 - S_4 & S_4 + S_6 \\ S_3 + S_5 & S_1 - S_5 + S_6 + S_7 \end{bmatrix}$$

$$S_1 = (A_{00} - A_{11}) * (B_{00} + B_{11})$$
$$S_2 = (A_{01} - A_{11}) * (B_{10} + B_{11})$$
$$S_3 = A_{11} * (B_{10} + B_{00})$$
$$S_4 = (A_{00} + A_{01}) * B_{11}$$
$$S_5 = (A_{10} + A_{11}) * B_{00}$$
$$S_6 = A_{00} * (B_{01} - B_{11})$$
$$S_7 = (A_{10} - A_{00}) * (B_{00} - B_{01})$$

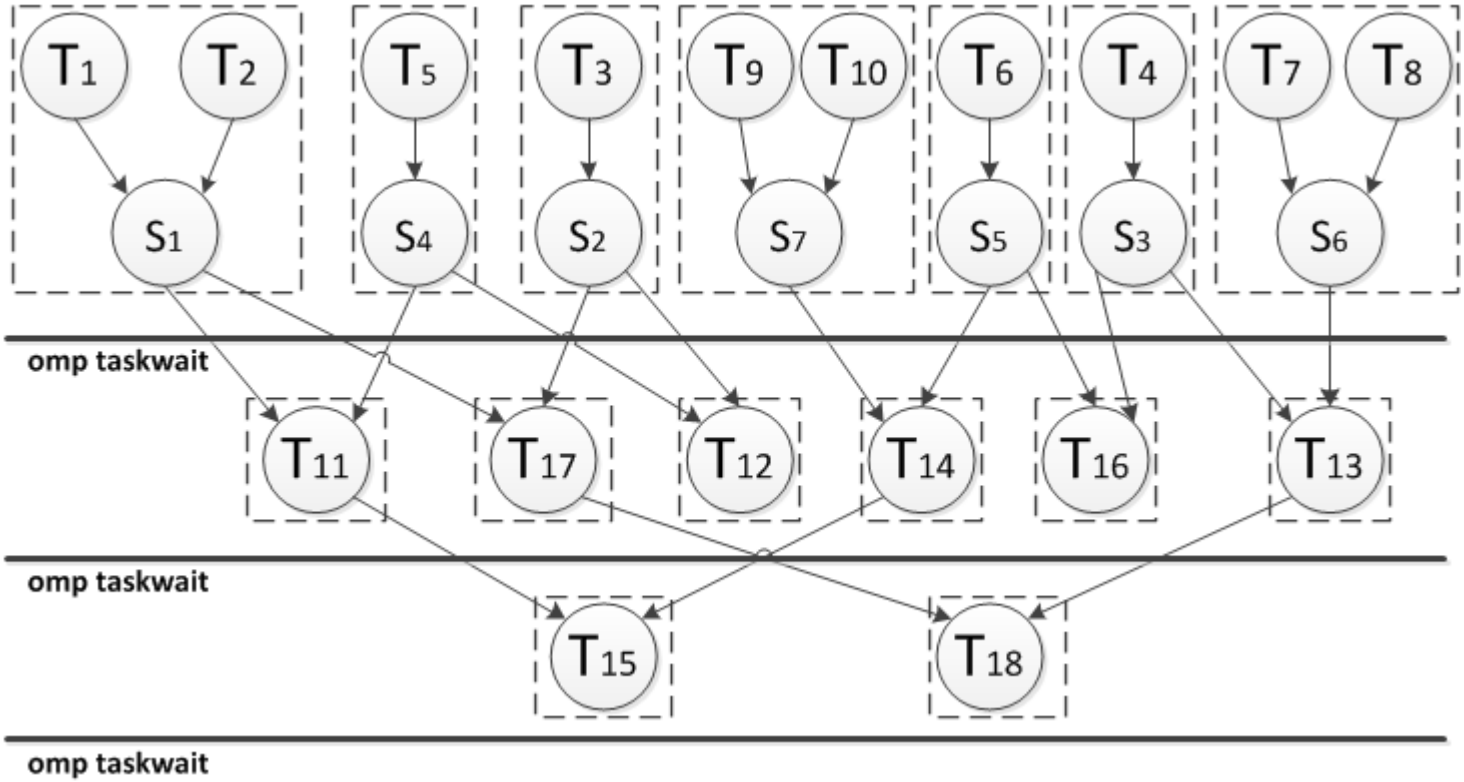
$$T_1 = A_{0,0} + A_{1,1}$$
$$T_2 = B_{0,0} + B_{1,1}$$
$$T_3 = A_{1,0} + A_{1,1}$$
$$T_4 = B_{0,1} - B_{1,1}$$
$$T_5 = B_{1,0} - B_{0,0}$$
$$T_6 = A_{0,0} + A_{0,1}$$
$$T_7 = A_{1,0} - A_{0,0}$$
$$T_8 = B_{0,0} + B_{0,1}$$
$$T_9 = A_{0,1} - A_{1,1}$$
$$T_{10} = B_{1,0} + B_{1,1}$$

$$S_1 = T_1 \cdot T_2$$
$$S_2 = T_3 \cdot B_{0,0}$$
$$S_3 = A_{0,0} \cdot T_4$$
$$S_4 = A_{1,1} \cdot T_5$$
$$S_5 = T_6 \cdot B_{1,1}$$
$$S_6 = T_7 \cdot T_8$$
$$S_7 = T_9 \cdot T_{10}$$

$$T_{11} = S_1 + S_4$$
$$T_{12} = S_2 + S_4$$
$$T_{13} = S_3 + S_6$$
$$T_{14} = S_7 - S_5$$
$$T_{16} = S_3 + S_5$$
$$T_{17} = S_1 - S_2$$

$$T_{15} = T_{11} + T_{14}$$
$$T_{18} = T_{13} + T_{17}$$

Task parallelism



Build and run

```
./build_x64.sh
./build_mic.sh

./run -t 512 -n 60 -s 8192 -r 1

./run -t 512 -n 120 -s 8192 -r 1

./run -t 512 -n 180 -s 8192 -r 1

./run -t 512 -n 240 -s 8192 -r 1
```

Intel(R) Math Kernel Library vs nested parallelism

If your application uses OpenMP threading, you may need to provide additional settings:

- Set the environment variable OMP_NESTED=true, or alternatively call omp_set_nested(1), to enable OpenMP nested parallelism.
- Set the environment variable MKL_DYNAMIC=false, or alternatively call mkl_set_dynamic(0), to prevent Intel MKL from dynamically reducing the number of threads in nested parallel regions.

https://software.intel.com/en-us/node/528380

https://software.intel.com/en-us/node/528547

mkl_set_num_threads

Intel MKL environment variable: MKL_NUM_THREADS

```
#include "mkl_service.h"

…

mkl_set_num_threads(4);

// Intel MKL uses up to 4 threads

my_compute_using_mkl();
```

mkl_set_num_threads_local

```
#include "mkl_service.h"
void my_compute( int nt )
{
        // save the Intel MKL number of threads
        int save = mkl_set_num_threads_local( nt );
// Intel MKL functions use up to nt threads on this thread
        my_compute_using_mkl();
        // restore the Intel MKL number of threads
        mkl_set_num_threads_local( save );
}
```

mkl_domain_set_num_threads

Intel MKL environment variable: MKL_DOMAIN_NUM_THREADS

```
#include "mkl_service.h"

mkl_domain_set_num_threads(4, MKL_DOMAIN_BLAS);

// Intel MKL BLAS functions use up to 4 threads

my_compute_using_mkl_blas();

// Intel MKL FFT functions use the default number of threads

my_compute_using_mkl_dft();
```

MKL_DOMAIN_ALL	All function domains
MKL_DOMAIN_BLAS	BLAS Routines
MKL_DOMAIN_FFT	non-cluster Fourier Transform Functions
MKL_DOMAIN_VML	Vector Mathematical Functions
MKL_DOMAIN_PARDISO	PARDISO

Nested parallelism

```
mkl_set_dynamic(0);
omp_set_nested(1);
set_num_threads(num_threads);
mkl_set_num_threads(mkl_num_threads);
strassen( … );
…
mkl_set_num_threads(num_threads * mkl_num_threads);
dgemm( … );
```

Parallel version with nested parallelism

- Folder 3

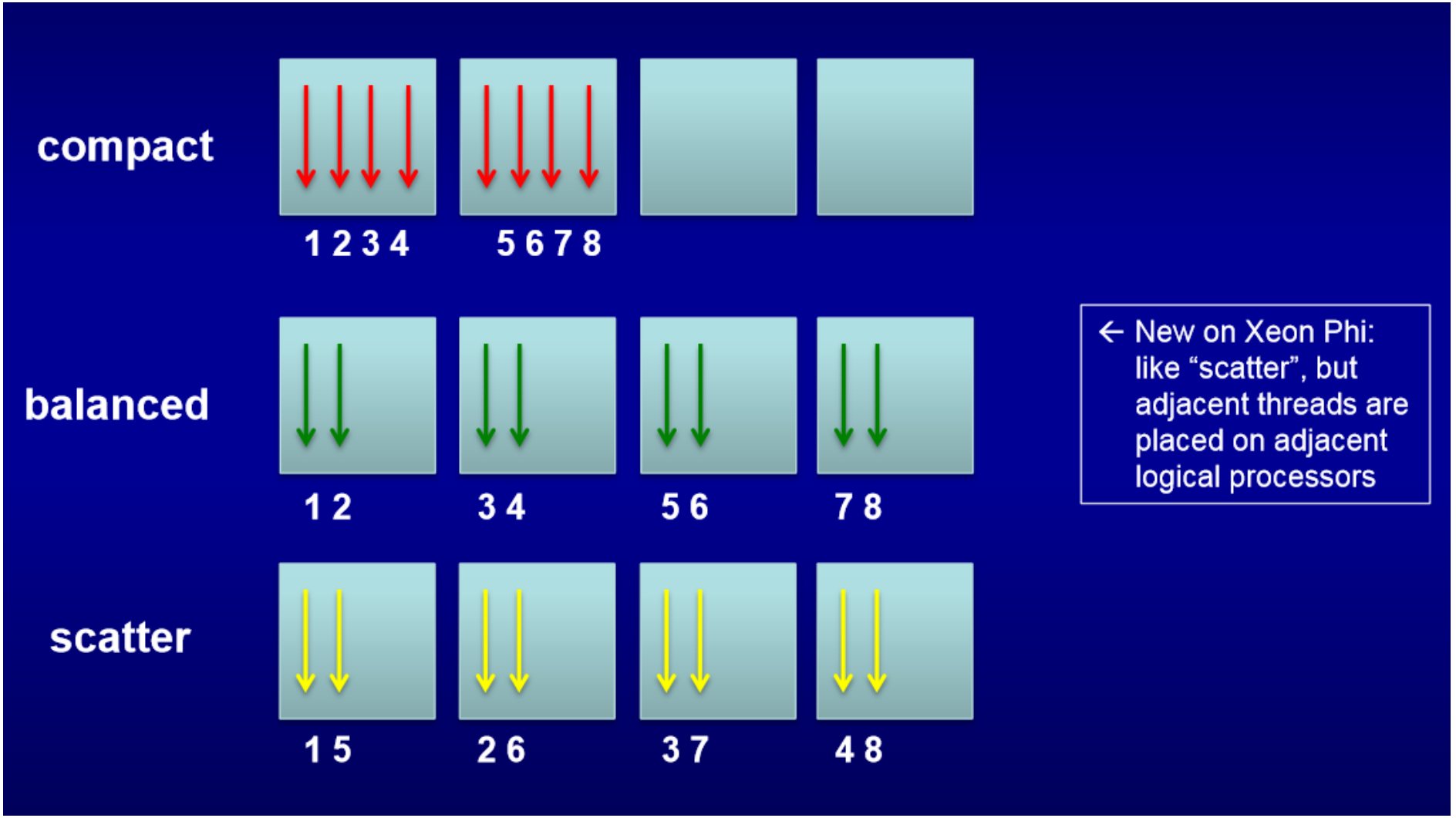
Build and run

```
./build_x64.sh
./build_mic.sh

./run -t 512 -n 60 -z 4 -s 8192 -r 1

./run -t 512 -n 120 -z 2 -s 8192 -r 1

./run -t 512 -n 180 -z 1 -s 8192 -r 1

./run -t 512 -n 240 -z 1 -s 8192 -r 1
```

OpenMP Thread Affinity Control

- https://software.intel.com/en-us/articles/openmp-thread-affinity-control
- Intel® Xeon Phi™ Coprocessor supports 4 thread contexts per core
- try different numbers of threads from $N-1$ threads to $4*(N-1)$ threads where N is the number of physical cores on the processor. Four simple experiments can be run: run the application with $(N-1)$ threads. Run with $2*(N-1)$, $3*(N-1)$ and $4*(N-1)$ to determine if the addition thread contexts give a performance benefit to your application.

KMP_AFFINITY distribution options



KMP_AFFINITY Examples

- `KMP_AFFINITY = granularity=fine,balanced`
- `KMP_AFFINITY = granularity=fine,compact`
- `KMP_AFFINITY = granularity=core,balanced`
- `KMP_AFFINITY=explicit,proclist=[1-240:1,0,241,242,243], granularity=fine`
- `KMP_AFFINITY=explicit,proclist=[0-243:2],granularity=fine`

Build and run

- `export KMP_AFFINITY=granularity=fine,scatter`
- `export KMP_AFFINITY=granularity=fine,balanced`
- `export KMP_AFFINITY=granularity=fine,compact`
- `export KMP_AFFINITY=granularity=core,balanced`
- `export KMP_AFFINITY=granularity=core,compact`

- `./run -t 512 -n 60 –z 4 -s 8192 -r 1`

KMP_PLACE_THREADS

value = (int ["C" | "T"] [delim] | delim) [int ["T"] [delim]] [int ["O"]];

Specifies the number of cores, with optional offset value and number of threads per core to use.

- "C" indicates Cores
- "T" indicates Threads
- "O" (letter O, not zero) is used to specify an Offset. Offset ignores granularity, Offset is the number of Cores to offset, starting from 0 "Core 0". Thus 1O would be the 2nd core in the package, aka "Core 1". Default is 0O
- Either cores or threads should be specified. If omitted, the default value is the available number of cores (threads).

KMP_PLACE_THREADS Examples

- 5C,3T,1O - use 5 cores with offset 1, 3 threads per core
- 5,3,1 - same as above
- 24 - use first 24 cores, 4 threads per core
- 2T - use all cores, 2 threads per core
- ,2 - same as above
- 3x2 - use 3 cores, 2 threads per core
- 4C,12O - use 4 cores with offset 12, all available threads per core

Extra tips

- Building Affinity Into an Executable
- Performance Tip: KMP_BLOCKTIME Parameter
- Performance Tip: -qopt-threads-per-core=1/2/3/4 (default is 4), gives a hint to the compiler about how many threads are likely to be running on the core for the application.
- Prefetch: https://software.intel.com/sites/default/files/managed/21/ea/5.3-prefetching-on-mic-5.pdf

Thank you for your attention

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