Application of Quantum Information Theory outside of Physics: Cognition and Decision Making

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Quantum(-like) operational representation of the process of decision making by cognitive systems

My talk is not about **quantum brain** in the spirit of R. Penrose and S. Hameroff. We do not try to reduce information processing by cognitive system to quantum **physical effects**.

The brain is a black box which works with information and probabilities in such a way that some features of processing cannot be described by classical theories. And there are a plenty of statistical data in cognitive psychology, game theory, decision making. One just has to understand its non-classicality, see:

A. Khrennikov, *Ubiquitous quantum structure: from psychology to finances*, Springer, Berlin-Heidelberg-New York, 2010.

In cognitive psychology such data is interpreted as pathological. We propose to consider data as simply nonclassical and try to apply the most well developed non-classical theories of information and probability, namely, based on the mathematical formalism of QM.

Law of total probability (LTP) .

Theorem in the Kolmogorov probability model. It is a consequence of additivity of probability and Bayes' formula for conditional probabilities.

$$p(B|A) = p(B \cap A)/p(A), p(A) > 0. \tag{1}$$

Consider two random variables $a = \pm 1, b = \pm 1.$ The b-variable describes decisions. So, we can make the decision $b = +1,$ "yes", or $b = -1,$ "no". The a-variable describes possible conditions, **contexts**, preceding the decision making.

For example, $a = +1$: the climate will change towards warming, $a = -1$: not; $b = +1$: to buy a property near sea, $b = -1$: not.

LTP *The prior probability to obtain the result, e.g.,* $\boldsymbol{b = +1}$ *for the random variable* $\boldsymbol{b}$ *is equal to the prior expected value of the posterior probability of* $\boldsymbol{b = +1}$ *under conditions* $\boldsymbol{a = +1}$ *and* $\boldsymbol{a = -1}$.

$$\boldsymbol{p(b = j) =}$$

$$\boldsymbol{p(a = +1)p(b = j|a = +1) + p(a = -1)p(b = j|a = -1),}$$

where $\boldsymbol{j = +1}$ *or* $\boldsymbol{j = -1}$.

LTP gives a possibility to predict the probabilities for the $\boldsymbol{b}$-variable on the basis of conditional probabilities and the $\boldsymbol{a}$-probabilities.

The cornerstone of Kolmogorov's approach is the postulation of a possibility to embed all complexes of conditions (contexts) preceding the decision making into one probability space. This postulated (!) embedding provides a possibility to apply to contexts the set-theoretical algebra, *Boolean algebra*, operations of intersection, union and complement.

Each mathematical model has a restricted domain of application; in particular, the Kolmogorov model of probability (cf. with the Euclidean model of geometry).

Two slit experiment and LTP

The $\boldsymbol{b}$-observable is the position of photon on the registration screen. To make the $\boldsymbol{b}$-variable discrete, we split the registration screen into two domains say $\boldsymbol{B_+}$ and $\boldsymbol{B_-}$. The $\boldsymbol{a}$-variable describes the slit which is used by a particle; say $\boldsymbol{a = +1}$ the upper slit and $\boldsymbol{a = -1}$ the lower slit. Consider three different experimental contexts:

$\boldsymbol{C}$: both slits are open. We can find $\boldsymbol{p(b = +1)}$ and $\boldsymbol{p(b = -1)}$ from the experiment as the frequencies of photons hitting the domains $\boldsymbol{B_+}$ and $\boldsymbol{B_-}$, respectively.

$\boldsymbol{C_+^a}$: only one slit, labeled by $\boldsymbol{a = +1}$, is open. We can find $\boldsymbol{p(b = j|a = +1), j = \pm 1}$.

$\boldsymbol{C_-^a}$: only one slit, labeled by $\boldsymbol{a = -1}$, is open. We can find $\boldsymbol{p(b = j|a = -1), j = \pm 1}$, the frequencies of photon hitting $\boldsymbol{B_+}$ and $\boldsymbol{B_-}$, respectively.

If we put these frequency-probabilities, collected in the three real experiment, we see that LTP is violated.

Can LTP be violated outside quantum physics? Why not!

What was the crucial probabilistic point of the previous analysis of the two slit experiment? Three experimental contexts $\boldsymbol{C}, \boldsymbol{C}^{\boldsymbol{a}}_{\pm}$ cannot be embedded in the same space $\boldsymbol{\Omega},$ since one cannot apply to the real physical situation the Boolean algebra.

In the philosophic terms this story is about the *principle of complementarity.* If we specify the slit, e.g., context $\boldsymbol{C}^{\boldsymbol{a}}_{+},$ we specify particle features of a quantum system. This destroys the context $\boldsymbol{C}$ describing the wave features (interference of two waves propagating through two open slits).

Historically: principle of complementarity was, in fact, borrowed by N. Bohr from psychologists. So, now we want to deliver it back to them but with the great mathematical apparatus.

Contextual viewpoint on violation of LTP

Given context C (a complex of conditions: physical, social, financial) and two dichotomous observables a and $b.$

These variables under context C have probabilities:

$$p(a = \pm 1|C), p(b = \pm 1|C);$$

here, e.g., $p(a = +1|C)$ is the probability that $a = +1$ under context (condition) $C.$

We emphasize that **context-conditioning** is not based on the Bayes' formula.

An important class of contexts is given by **selection contexts** corresponding to conditioning upon values of some variable.

Take a variable, say $a,$ taking two values $a = \pm 1.$ Consider two contexts C_+^a : the condition that a takes the value $a = +1,$ and C_-^a : the condition that a takes the value $a = -1.$

For example, a is a question asked to a group (ensemble) of people. Here contexts $C^a_\pm$ have the ensemble representation: the C^a_+ by the ensemble of people who replied "yes" and the C^a_+ by those who replied "no."

Take variable b. Under selection-contexts $C^a_\pm$, perform the b-measurement and obtain contextual (conditional) probabilities: $p(b = \pm 1 | C^a_+), p(b = \pm 1 | C^a_-)$. To make notation closer to the standard one, we set

$$p(b = \pm 1 | C^a_+) \equiv p(b = \pm 1 | a = +1), \ldots$$

If LTP does not hold true (as a consequence of **multi-contextuality with complementary contexts**), then the left-hand side of is not equal to the right-hand side. We call this difference **interference term** by analogy with quantum mechanics in which the LTP is violated inducing so called interference term. In contextual notations the interference terms are given by

$$\delta(b = \pm 1 | C) = p(b = \pm 1 | C) - \sum_\alpha p(a = \alpha | C) p(b = \pm 1 | a = \alpha),$$

This definition can be rewritten as LTP perturbed by the interference terms:

$$\boldsymbol{p(b = \pm 1|C) = \sum_{\alpha} p(a = \alpha|C)p(b = \pm 1|a = \alpha) + \delta_{\pm},}$$

where $\boldsymbol{\delta_{\pm} = \delta_{\pm}(C).}$ By analogy with QM we select normalization:

$$\boldsymbol{\lambda_{\pm} = \delta_{\pm}/2\sqrt{\Pi_{\pm}},} \tag{2}$$

where

$$\boldsymbol{\Pi_{\pm} \equiv \Pi_{\pm}(C) = \prod_{\alpha} p(a = \alpha|C)p(b = \pm 1|a = \alpha)} \tag{3}$$

Thus LTP with interference terms can be written as

$$\boldsymbol{p(b = \pm|C) = \sum_{\alpha} p(a = \alpha|C)p(b = \pm 1|a = \alpha) + 2\lambda_{\pm}\sqrt{\Pi_{\pm}}.}$$

If the absolute values of the normalized interference term $\boldsymbol{\lambda_q}$ is less than 1 (for some $\boldsymbol{q = \pm 1}$), we can find such an angle $\boldsymbol{\theta_q}$ that

$$\boldsymbol{\lambda_q = \cos \theta_q;} \tag{4}$$

In the trigonometric case we have the following LTP with the interference term:

$$\boldsymbol{p(b = \pm|C) = \sum_{\alpha} p(a = \alpha|C)p(b = \pm 1|a = \alpha) + 2\cos\theta_{\pm}\sqrt{\Pi_{\pm}}.}$$

This sort of interference between probabilities can be easily derived in the standard formalism of complex Hilbert space used in QM. Contexts are represented by quantum states (normalized vectors or more generally density operators), observables $\boldsymbol{a}$ and $\boldsymbol{b}$ by Hermitian operators or more generally POVMs, probabilities are defined by Born's rule.

We remark that $\boldsymbol{\lambda_q = 0}$ for any quantum state iff POVMs commute.

In the case of observables given by projection valued measures the interference coefficients $\boldsymbol{\lambda_q \leq 1.}$ For POVMs they can exceed 1. (see Foundations of Physics 34 (4), 689-704 (2004).)

Question: let there are given arbitrary probabilistic data $p(b = \pm|C), p(a = \pm 1|C), p(b = \pm 1|a = \pm 1)$. Can we find two POVMs a and b and density operator reproducing these data?

Modification of the question 2: suppose that the state is given by

$$\psi = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle).$$

Multidimensional case?

Partial solutions: **A. Khrennikov, Contextual Approach to Quantum Formalism. Springer, 2009.**

Interference effects in social science

Savage Sure Thing Principle

Savage, L.J. **The foundations of statistics. New York: Wiley and Sons (1954).**

STP *If you prefer prospect* $\boldsymbol{b}_+$ *to prospect* $\boldsymbol{b}_-$ *if a possible future event* $\boldsymbol{A}$ *happens* ($\boldsymbol{a} = +1$), *and you prefer prospect* $\boldsymbol{b}_+$ *still if future event* $\boldsymbol{A}$ *does not happen* ($\boldsymbol{a} = -1$), *then you should prefer prospect* $\boldsymbol{b}_+$ *despite having no knowledge of whether or not event* $\boldsymbol{A}$ *will happen.*

Savage's illustration refers to a person deciding whether or not to buy a certain property shortly before a presidential election, the outcome of which could radically affect the property market. "Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event will obtain".

Rationality

A decision maker has to be *rational.* **Thus the STP was used as one of foundations of** *rational decision making* **and rationality in general. It plays an important role in economics in the framework of Savage's** *utility theory.*

Savage's STP is a simple consequence of LTP.

LTP: Bayes conditioning + additivity of probability.

Behavioral games: Prisoner's dilemma

Two suspects, A and B, are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal:

a). If one testifies for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives the full 10-year sentence.

b). If both stay silent, both prisoners are sentenced to only six months in jail for a minor charge.

c). If each betrays the other, each receives a two-year sentence.

Each prisoner must make the choice of whether to betray the other or to remain silent. However, neither prisoner knows for sure what choice the other prisoner will make. So this dilemma poses the question:

How should the prisoners act?

Rational prisoners, i.e., prisoners who proceed on the basis of Savage STP, should always (both) select the strategy to betray, i.e., to cooperate with police.

Violation of rationality in the experiments

The following mental contexts are involved in PD:

Context C, representing the situation when a player has no idea about the planned action of the other player, "uncertainty context."

Context C_+^a, representing the situation when the B-player supposes that A will stay silent (cooperate), and context C_-^a, when B supposes that A will betray (compete).

Croson, R. The disjunction effect and reasoning-based choice in games. Organizational Behavior and Human Decision Processes 80, 118-133 (1999).

Another version: Alice is informed about the decision of Bob.

Shafir, E. and Tversky, A. Thinking through uncertainty: nonconsequential reasoning and choice. Cognitive Psychology 24, 449-474 (1992)

Tversky, A. and Shafir, E. The disjunction effect in choice under uncertainty. Psychological Science, 3, 305-309 (1992)

We define dichotomous variables a and b corresponding to *actions* of players A and B : $a = +1$ if A chooses to cooperate and $a = -1$ if A chooses to compete; b values are defined in the same way.

In the Shafir–Tversky PD experiment: $p(b = -1|C) = 0.63$ and hence $p(b = +1|C) = 0.37$;

$p_{--} = 0.97,\ p_{-+} = 0.03;\ \ p_{+-} = 0.84,\ p_{++} = 0.16.$

Matrix of transition probabilities

$$p^{b|a} = \begin{pmatrix} 0.16 & 0.84 \\ 0.03 & 0.97 \end{pmatrix}.$$

It is *stochastic, but not doubly stochastic!*

Here a-probabilities are equal: they were produced simply by a random generator imitating the first play of the gamble.

Simple arithmetic calculations give

$$\boldsymbol{\delta_+ = -0.28, \quad \lambda_+ = -0.44, \delta_- = 0.28, \quad \lambda_- = 0.79.}$$

The coefficients of interference are nonzero! Thus the probabilistic data are nonclassical. These coefficients are bounded by 1. Thus, a nonclassical version of LTP (with the trigonometric interference) holds true. Here probabilistic phases

$$\boldsymbol{\theta_+ = 2.03, \theta_- = 0.66.}$$

Order Effects

In a typical opinion-polling experiment, a group of participants is asked one question at a time, e.g., a =“Is Bill Clinton honest and trustworthy?” and then b =“Is Al Gore honest and trustworthy?”

The joint probability distribution is found $p(a = \alpha, b = \beta), \alpha, \beta = \pm 1$. Then these questions are asked in the opposite order, the joint probability distribution is found $p(b = \beta, a = \alpha), \alpha, \beta = \pm 1$. And these distributions do not coincide.

Such noncommutative effect cannot be represented in the Kolmogorov model, by representing questions by random variables. In the quantum formalism we can easily model this effect by using non-commutative POVMs.

(a-a)-problem.

We remark that in fact we have to use projection valued measures, since if, e.g., the value $a = +1$ was received and the question a asked the second time, the answer $a = +1$ is obtained with probability 1. The same is happens for the b-question.

Repeatable measurement implies that, in fact, POVM is a projector valued measure! In the finite dimensional case!

Buscemi, F., D' Ariano, G. M. and Perinotti, P.: There exist nonorthogonal quantum measurements that are perfectly repeatable. Phys. Rev. Lett. 92, 070403-1 - 070403-4 (2004).

Thus, if we want to describe the Clinton-Gore experiment in the quantum-like manner we have to represent the questions A and B by projection-type observables.

(a-b-a)-problem.

However, the real situation is more complicated. Even in the sequence (a-b-a), if the first result was $a = +1$, then for any result $b = \beta$ the result of the second a-measurement is again $a = +1$ with probability 1. It is possible to show that this is possible only if a and b are projector valued measures and they commute. However, commutativity is incompatible with order effect.

A. Khrennikov, Basieva, I., Dzhafarov, E.N., Busemeyer, J.R. (2014). Quantum Models for Psychological Measurements : An Unsolved Problem. PLoS ONE. 9. Article ID: e110909.

For atomic instruments, this was proven in:

I. Basieva, A. Khrennikov, On a possibility to combine the order effect with sequential reproducibility for quantum measurements. *Found. Phys.* 45, N 10, 1379-1393 (2015).

For non-atomic, we do not know, neither in the infinite-dimensional case.

Measurement problem in decision making

Nevertheless, majority of physicists are fine with collapse of the wave function.

In cognitive community there is common opinion that the mental state dynamics is continuous and selection of different alternatives in the process of decision making cannot be modeled by the collapse type process.

One of attempts to solve the measurement problem is based on consideration of measurement process as the decoherence process, W. Zurek and recently G. Lindblad. In the limit $t \to \infty$ the state $\rho(t)$ approaches the state ρ_{out} which is diagonal in "pointer basis".

Decision making as decoherence

The model is pure informational, both a "quantum-like system"and bath are represented by their states, ψ and ϕ, we are not interested in their physical realizations. In the PD, the ψ represents Bob's possible decisions and ϕ the "information bath"having some degree of relevance to this concrete problem; in particular, Bob's recollections about Alice. Then we apply theory of open quantum systems and Gorini-Kossakowski-Sudarshan-Lindblad dynamics to model experimental data starting from the state of complete uncertainty, $\psi = (|-\rangle + |+\rangle)/\sqrt{2}$.

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Classical Aumann theorem

Mutual knowledge: **everybody in a group of people is aware about some fact or event.**

Common Knowledge: **Alice and Bob knows about an event E and Alice knows that Bob knows about E and so on...**

The celebrated Aumann theorem states that if two agents have common priors, and their posteriors for a given event E are common knowledge, then their posteriors must be equal;

Agents with the same priors and common knowledge about posteriors cannot agree to disagree.

Criticized assumptions:

a). common priors, but typically it is justified – as the result of information exchange.

b). common knowledge about posteriors, but again Aumann's statement can be violated even in situations, where this assumption is valid.

This situation is disturbing and the debate about possible sources of violation Aumann's theorem are continued.

We point to an implicit assumption of Aumann:

Agents are rational, where rationality is understood as the use of Bayes' rule to update probabilities.

Agents may update probabilities with schemes different from CP. QP update is a possible math formalism describing non-Bayesian updates. Such agents may agree to disagree; even with common priors and common knowledge.

Classical probabilistic approach to common knowledge

Agents, call them $i = 1, 2, ..., N$. These individuals are about to learn the answers to various multi-choice *questions*, to make observations.

Classical probability space (Ω, F, P). Events are subsets of Ω.

Each agent creates its information representation for possible *states of the world* based on its own possibilities to perform measurements, "to ask questions to the world."

The representations are given by disjoint partitions of Ω : $P^{(i)} = (P_j^{(i)})$, where

$$\cup_j P_j^{(i)} = \Omega \text{ and } P_j^{(i)} \cap P_k^{(i)} = \emptyset, j \neq k.$$

Thus an agent cannot get to know the state of the world ω precisely; she can only get to know to which element of its information partition $P_j^{(i)} = P_j^{(i)}(\omega)$ this ω belongs.

Definition. The agent i knows an event E in the state of the world ω if

$$P_j^{(i)}(\omega) \subset E. \tag{5}$$

It is assumed that on Ω there is defined a probability p, the common prior of all agents.

We now present the definition of common knowledge for two agents:

CN An event E is common knowledge at the state of the world ω if 1 knows E, 2 knows E, 1 knows 2 knows E, 2 knows 1 knows E, and so on.

Denote the set of all states of the world for which E is common knowledge by the symbol κE.

Aumann: for each agent i, the set κE can be represented (in the case $\kappa E \neq \emptyset$) in the form:

$$\kappa E = \cup_m P^{(i)}_{j_m}. \tag{6}$$

We also remark that the conditional probability $\mathrm{q}_i(\omega)$ that i-th agent assigns to some event E is defined to be the same for all states of the world ω in a given element of partition,

$$\mathrm{q}_i(\omega) = p(E \cap P(\omega))/p(P(\omega)).$$

Thus, in fact,

$$\mathrm{q}_i(\omega) \equiv \mathrm{q}_{ik},$$

where $\omega \in P_k^{(i)} = P(\omega)$.

Aumann's theorem states that if both

$$\mathrm{q}_1(\omega) = q_1 \text{ and } \mathrm{q}_2(\omega) = q_2 \tag{7}$$

are common knowledge and prior probabilities are the same, then necessarily $q_1 = q_2$ - simply because

$$q_i = p(E|\kappa C_{q_1 q_2}) = p(E \cap \kappa C_{q_1 q_2})/p(\kappa C_{q_1 q_2}), \tag{8}$$

where $C_{q_1 q_2}$ is the event (7)).

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In order to avoid confusion concerning conditioning on *posterior probabilities being common knowledge*, **we can reformulate Aumann's theorem as: given the common priors, posterior probabilities may be common knowledge only when they are equal.**

Quantum(-like) formalization of common knowledge

By Birkhoff-von Neumann *events* are represented as orthogonal projectors.

For an orthogonal projector P, we set $H_P = P(H)$, its image, and vice versa, for subspace L of H, the corresponding orthogonal projector is denoted by the symbol P_L.

The set of orthogonal projectors is a *lattice* with the order structure: $P \leq Q$ iff $H_P \subset H_Q$ or equivalently, for any $\psi \in H$, $\langle\psi|P\psi\rangle \leq \langle\psi|Q\psi\rangle$. This lattice is called quantum logic.

Quantum representation of the states of the world

In our model the states of the world are given by pure states.

Definition. For the state of the world ψ, an event P *occurs* (takes place with probability 1) if ψ belongs to H_P.

Questions posed by agents are mathematically described by self-adjoint operators, say $A^{(i)}$:

$$A^{(i)} = \sum_j a_j^{(i)} P_j^{(i)}, \tag{9}$$

where (a_j) encode possible answers to the question of the ith agent.

For any agent i, $(P_j^{(i)})$ is a "disjoint partition of unity":

$$\bigvee_k P_k^{(i)} = I,\ P_k^{(i)} \wedge P_m^{(i)} = 0, k \neq m. \tag{10}$$

This spectral family is *information representation* **of the world by the ith agent.**

Opposite to the classical case, in general the state of the world ψ need not belong to any concrete subspace $H_{P_k^{(i)}}$.

Nevertheless, for any pure state ψ, there exists the minimal projector $Q_\psi^{(i)}$ of the form $\sum_m P_{j_m}^{(i)}$ such that $P_\psi \leq Q_\psi^{(i)}$.

The projector $Q_\psi^{(i)}$ represents the ith agent's knowledge about the ψ-world. We remark that $p_\psi(Q_\psi^{(i)}) = 1$.

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Knowing events: quantum representation

Consider the system of projectors $\tilde{P}^{(i)}$ consisting of sums of the projectors from $P^{(i)}$:

$$\tilde{P}^{(i)} = \{P = \sum_m P_{j_m}^{(i)}\}. \tag{11}$$

Then

$$Q_\psi^{(i)} = \min\{P \in \tilde{P}^{(i)} : P_\psi \leq P\}. \tag{12}$$

Definition. For the ψ-state of the world and the event E, the ith agent knowns E if

$$Q_\psi^{(i)} \leq E. \tag{13}$$

Common knowledge: quantum representation

We use the standard definition of common knowledge.

We recall that in the classical case, for each event E, there is considered the set of all states of the world for which E is common knowledge. It is denoted by the symbol κE.

This definition is naturally generalized to the quantum case. It can be proven that the set κE of quantum states of the world is a linear subspace of the state space H. Hence, we can define the projector on it; it is denoted by the same symbol.

Similar to the set-theoretic framework, we introduce the system of projectors $\tilde{P} = \cap_i \tilde{P}^{(i)}$. We remark that (by definition) a projector $P \in \tilde{P}$ if and only if, for each $i = 1, ..., N$, it can be represented in the form

$$P = \sum_m P^{(i)}_{j_m}. \tag{14}$$

Lemma 1. If $\kappa E \neq 0$, then, for any agent i, it can be represented as the sum of orthogonal projectors:

$$\kappa E = \sum_m P^{(i)}_{j_m}. \tag{15}$$

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Quantum state update: projection postulate

There are given a state $\boldsymbol{\rho}$ and an observable $\boldsymbol{A = \sum_i a_i P_i}$. Then

$$\boldsymbol{p_\rho(a_i) = \mathrm{Tr}\rho \mathrm{P_i}}. \tag{16}$$

However, if after measurement of the $\boldsymbol{A}$-observable one plans to perform measurement of another observable $\boldsymbol{B = \sum_i b_i P'_i)}$, then one needs to know even the output state:

$$\boldsymbol{\rho_{a_i} = \frac{P_i \rho P_i}{\mathrm{Tr} \mathrm{P_i} \rho \mathrm{P_i}}}. \tag{17}$$

This nothing else than **the quantum version of the classical rule for probability update.** But here we update not the prior probability, but the *prior state*.

For the $\boldsymbol{B}$-measurement following the $\boldsymbol{A}$-measurement, this state plays the same role as the state $\boldsymbol{\rho}$ played for the $\boldsymbol{A}$-measurement. In particular, by applying the Born rule once again we obtain:

$$\boldsymbol{p_{\rho_{a_i}}(b_j) = \mathbf{Tr}\rho_{\mathrm{a_i}}\mathrm{P}'_{\mathrm{j}} = \frac{\mathbf{Tr}\mathrm{P_i}\rho\mathrm{P_i}\mathrm{P}'_{\mathrm{j}}}{\mathbf{Tr}\mathrm{P_i}\rho\mathrm{P_i}}}. \tag{18}$$

In quantum theory this probability is treated as the **conditional probability** $\boldsymbol{p_\rho(P'_j|P_i) \equiv p_\rho(B = b_j|A = a_i)}$.

Quantum(-like) viewpoint on the Aumann's theorem

Common prior assumption

Suppose now that both agents assign to possible states of the world the same quantum probability distribution given by the density operator ρ, a priori state. Thus they do not know exactly the real state of the world (the latter is always a pure state) and in general a possible state of the world appears for them as a mixed quantum state ρ.

Disagree from quantum(-like) interference

Now we repeat classical Aumann's scheme of (dis)agreement on disagree. The only difference from the classical case is that the agents use another (non-Bayesian) rule to update probabilities.

Consider some event E. The agents assign to it probabilities after conditioning ρ on the answers to their questions (on their information representations of the world):

$$\mathrm{q}_{ik} = p_\rho(E|P_k^{(i)}) = \frac{\mathrm{Tr} \mathrm{P}_\mathrm{k}^{(\mathrm{i})} \rho \mathrm{P}_\mathrm{k}^{(\mathrm{i})} \mathrm{E}}{\mathrm{Tr} \mathrm{P}_\mathrm{k}^{(\mathrm{i})} \rho \mathrm{P}_\mathrm{k}^{(\mathrm{i})}}. \tag{19}$$

Thus q_{ik} is the probability which the ith agent would assign to the event E under condition that she gets the answer $a_k^{(i)}$ to her question-observable $A^{(i)}$.

For each i, consider the event

$$C_{q_i} \equiv \{\mathrm{q}_{ik} = q_i\}$$

that after observing her result, the ith agent assigned the value q_i (i.e., i observed one of the values $a_k^{(i)}$ leading to the probability $\mathrm{q}_{ik} = q_i$).

We also consider the event

$$C_{q_1 \ldots q_N} = \{\mathrm{q}_{1k} = q_1, \ldots, \mathrm{q}_{1k} = q_N\}$$

that after observing their results the agents assigned the values $q_i \in V_i, i = 1, 2, \ldots, N$, to the event E.

Try to repeat the standard proof of the Aumann theorem. By Lemma 1 the common knowledge projector (for the event $C_{q_1 \ldots q_N}$) can be represented as

$$\kappa C_{q_1 \ldots q_N} = \sum_j P_{k_j}^{(i)}, i = 1, \ldots, N.$$

For each such $P_{k_j}^{(1)}, .., P_{k_j}^{(N)}$, we have

$$p_\rho(E|P_{k_j}^{(1)}) = q_1, ..., p_\rho(E|P_{k_j}^{(N)}) = q_N. \tag{20}$$

Consider now the conditional probability:

$$p_\rho(E|\kappa C_{q_1...q_N}) = \frac{\mathrm{Tr}\kappa \mathrm{C}_{\mathrm{q_1...q_N}}\rho\kappa \mathrm{C}_{\mathrm{q_1...q_N}}\mathrm{E}}{\mathrm{Tr}\kappa \mathrm{C}_{\mathrm{q_1...q_N}}\rho\kappa \mathrm{C}_{\mathrm{q_1...q_N}}}.$$

By using representation given by Lemma 1 we obtain

(21)
$$p_\rho(E|\kappa C_{q_1\dots q_N}) = \frac{1}{\mathrm{Tr}\rho\kappa\mathrm{C}_{\mathrm{q_1\dots q_N}}}\Big(\sum_j \mathrm{Tr}\mathrm{P}^{(i)}_{\mathrm{k_j}}\rho\mathrm{P}^{(i)}_{\mathrm{k_j}}\mathrm{E}+\sum_{\mathrm{j\neq m}}\mathrm{Tr}\mathrm{P}^{(i)}_{\mathrm{k_j}}\rho\mathrm{P}^{(i)}_{\mathrm{k_m}}\mathrm{E}\Big).$$

By using (20) the first (diagonal) sum can be written as

$$\frac{1}{\mathrm{Tr}\rho\kappa\mathrm{C}_{\mathrm{q_1\dots q_N}}}\sum_j \frac{\mathrm{Tr}\mathrm{P}^{(i)}_{\mathrm{k_j}}\rho\mathrm{P}^{(i)}_{\mathrm{k_j}}\mathrm{E}}{\mathrm{Tr}\rho\mathrm{P}^{(i)}_{\mathrm{k_j}}}\mathrm{Tr}\rho\mathrm{P}^{(i)}_{\mathrm{k_j}} = \frac{\mathrm{q_i}}{\mathrm{Tr}\rho\kappa\mathrm{C}_{\mathrm{q_1\dots q_N}}}\mathrm{Tr}\sum \rho\mathrm{P}^{(i)}_{\mathrm{k_j}}$$

$$= \frac{q_i}{\mathrm{Tr}\rho\kappa\mathrm{C}_{\mathrm{q_1\dots q_N}}}\mathrm{Tr}\rho\sum \mathrm{P}^{(i)}_{\mathrm{k_j}} = \mathrm{q_i}.$$

In the absence of the off-diagonal term in (21) we get (cf. (8)):

(22)
$$p_\rho(E|\kappa C_{q_1\dots q_N}) = q_i,$$

i.e., $q_1 = \dots = q_N$. This corresponds to the classical case.

However, in general the off-diagonal term does not vanish – this is *the interference type effect.*

Thus *agents processing information in the quantum logic framework can agree on disagree.*

Although the probabilities are not equal, it is useful to know the degree of mismatching between them and the quantum formalism provides such information in the form of the interference term.

Theorem 1. *Let the assumption of common prior holds. Then:*

$$q_i - q_s = \frac{1}{\mathrm{Tr}\rho\kappa C_{q_1 \ldots q_N}} \Big(\sum_{j \neq m} \mathrm{Tr} P^{(i)}_{k_j} \rho P^{(i)}_{k_m} E - \sum_{j \neq m} \mathrm{Tr} P^{(s)}_{k_j} \rho P^{(s)}_{k_m} E \Big). \tag{23}$$

Sufficient condition for validity of Aumann theorem is that

$$\sum_{j \neq m} (P_{k_j}^{(i)} \rho P_{k_m}^{(i)} - P_{k_j}^{(s)} \rho P_{k_m}^{(s)}) = 0.$$

Related to state based commutativity?

Sufficient condition, event based commutativity:

$$\sum_{j \neq m} (P_{k_j}^{(i)} E P_{k_m}^{(i)} - P_{k_j}^{(s)} E P_{k_m}^{(s)}) = 0.$$

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